

STATEMENT REPORT

under the procedure for acquisition of the the scientific degree "Doctor of Science"

by candidate Associate Professor Dr. Borislav Radkov Draganov,

of the DSci Thesis entitled: "Simultaneous Approximation by the Bernstein Operator",

In the Scientific field: 4. Natural Sciences, Mathematics and Informatics

Professional field: 4.5. Mathematics

Department „Mathematical Analysis”,

Faculty of Mathematics and Informatics (FMI), Sofia University “St. Kl. Ohridski” (SU),

The statement report has been prepared by: Prof. DSci Geno Nikolov, FMI, SU

as a member of the scientific jury for the defense of this DSci thesis according to Order № ПД

38-627 / 28.11.2023 г.. of the Rector of the Sofia University.

1. General characteristics of the dissertation thesis and the presented materials

The presented DSci Thesis is in English with the title *Simultaneous approximation by the Bernstein operator*, contains 178 pages and includes an introduction, 6 chapters and a list of 100 cited publications. The candidate Associate Professor Dr. Borislav Draganov has submitted all the documents required by the procedure, including: abstract (in Bulgarian and English), creative autobiography, diplomas for higher education and for the acquisition of PhD Degree, evidence for meeting the minimum national requirements under Art. 2b, para. 2 and 3 of ADASRB*, declaration concerning the authorship of the submitted DSci Thesis, declaration by Assoc. Prof. Ivan Gadjev for equal contribution of the candidate in their joint article, reference for citations of the articles whose content is included in the dissertation.

2. Short CV and personal impressions of the candidate

Dr. Borislav Draganov graduated from Faculty of Mathematics and Informatics of Sofia University (in what follows abbreviated to FMI and SU, respectively) in 1998. During the period 1999 – 2002 he was a PhD student at FMI with scientific supervisor Prof. Dr. Kamen Ivanov, and in 2002 acquires PhD Degree for a dissertation entitled *A new method for the characterization of K-functionals and application in approximation theory*. My acquaintance with Dr. Draganov dates from this period. His research interests are mainly in the field of approximation theory, and in particular, the establishment of sharp estimates for the rate of approximations by linear and non-linear operators in various function spaces, Fourier analysis and its applications, approximations of functions by information of discrete type, etc. He is also interested in history of mathematics, philosophy of science and physics. I am well aware of Dr. Draganov's scientific interests and publications also thanks to the fact that I was a reviewer for the competition by which he was selected as an Associate Professor in the Mathematical Analysis Department of FMI - SU in 2011. Dr. Draganov has been a lecturer at FMI-SU since 2002 as Assistant/Head Assistant (2002-2011), and as Associate Professor since October 2011 so far. In parallel with this, he worked at Institute of Mathematics and Informatics (IMI) of the Bulgarian Academy of Sciences in a position of Mathematician (2004-2007), Research Fellow (2007-2012). In July 2012 he has been selected as an Associate Professor in the "Mathematical Modeling and Numerical Analysis" section at IMI.

3. Content analysis of the scientific and applied achievements of the candidate, contained in the presented PhD thesis and the publications to it, included in the procedure

In 1912 Sergey Natanovich Bernstein introduced an operator which has been applied to prove the fundamental result in approximation theory that any continuous function on a finite closed interval can be uniformly approximated to arbitrary prescribed accuracy by an algebraic polynomial. Although numerous approximation operators have been (and continue to be) proposed and studied since then, the Bernstein operator remains the most popular and subject of intensive research even today, as evidenced by the numerous publications on it.

The subject of the dissertation is the simultaneous weighted approximation of functions and their derivatives by the Bernstein operator and its modifications (iterated Boolean sums and integer variants). Below I briefly describe the content of the dissertation.

In the Introduction, direct and converse estimates for unweighted approximation by the Bernstein operator are presented in terms of the classical moduli of continuity (smoothness) with uniform step or with step controlled by the function $\varphi(x) = \sqrt{x(1-x)}$. The author states the important result of Voronovskaya asserting the bounded rate of convergence of approximation to the Bernstein operator, and points out to its property to simultaneously approximate functions and their derivatives.

In Chapter 1 the author lists basic properties of the moduli of smoothness and K-functionals, defines their weighted variants, and states the existence of equivalence between smoothness moduli and appropriate K-functionals. In Chapter 2, Dr. Draganov proves some embedding inequalities, viz. estimates for the norms of intermediate derivatives (a classical example is the Landau-Kolmogorov inequality (2.1), but here specific variants with Jacobi weights and differential operators in terms of $Df = \varphi^2 f''$ is considered). These two chapters are of a preparatory nature, and further on, in the proofs of main results in the dissertation, in order to avoid the repetition of analogous reasoning, the author refers to them.

In Chapter 3, Dr. Draganov proves a direct estimate for the simultaneous approximation by Bernstein polynomials in uniform norm with Jacobi weights (Theorem 3.3). This estimate is with the K-functional

$$K_s^D(f, t)_w = \inf_{g \in C^{s+2}[0,1]} \left\{ \|w(f - g^{(s)})\| + t \|w(Dg)^{(s)}\| \right\},$$

and in Theorem 3.5 the author proves that it can be replaced by a sum of simpler characteristics of the approximated function. Draganov shows that his Jackson-type estimate is exact by proving the corresponding strong converse inequalities in Theorem 3.8. At the end of Chapter 3, using the relation between the Bernstein and Kantorovich operators, Draganov proves an analogous characterization for the rate of approximation of the latter operator.

Chapter 4 is devoted to the simultaneous approximation by the iterated Boolean sums of the Bernstein operator, $B_{r,n} := I - (I - B_n)^r$, where I is the identity operator and

$r \in \mathbb{N}$. For the convergence rate of the simultaneous uniform approximation by these operators with Jacobi weights, Dr. Draganov proves in Theorem 4.3 an upper estimate with the K-functional

$$K_{r,s}^D(f, t)_w := \inf_{g \in C^{2r+s}[0,1]} \left\{ \|w(f - g^{(s)})\| + t \|w(D^r g)^{(s)}\| \right\}$$

(for the approximation of the s -th derivative), showing again that it can be replaced by equivalent simpler characteristics (Theorems 4.4 - 4.8). The author demonstrates the sharpness of the direct estimate in Theorem 4.3 by proving a strong converse inequality in Theorem 4.10. Finally, from the results about the iterated Boolean sums of Bernstein operators, Dr. Draganov derives analogous direct and converse inequalities for the simultaneous approximation by iterated Boolean sums of the Kantorovich operator.

In Chapter 5, simultaneous approximations by two modifications of Bernstein polynomials, specifying polynomials with integer coefficients, are considered. One of them, \tilde{B}_n , has been introduced by Kantorovich in 1931, and Draganov proposes another one, denoted by \hat{B}_n . It should be mentioned that the operators \tilde{B}_n and \hat{B}_n are neither linear nor continuous, moreover, the operator $\tilde{B}_n: C[0,1] \rightarrow C[0,1]$ is not bounded. The requirement to approximate by polynomials with integer coefficients leads to additional restrictions on the set of functions to be approximated: some of them natural (to assume integer values at the ends of the interval) and others unexpected (derivatives vanish at the ends of the interval, inequalities for "tangents"). Under such assumptions, Dr. Draganov proves in Theorems 5.1 and 5.4 direct estimates for the approximation by these operators, moreover, he shows that the additional restrictions imposed on the set of the functions being approximated are necessary. Draganov proves in Theorem 5.5 weak inverse relations complementary to Theorems 5.1 and 5.4. Again, as in the previous two chapters, Draganov makes use of the relation between the Bernstein and Kantorovich operators to define an integer version of the latter and prove a direct estimate for the simultaneous approximations by them.

In Chapter 6 Draganov studies the approximation of the operator $\mathcal{D}f(x) = \frac{1}{2}\varphi^2(x)f''(x)$ by the Voronovskaya operator $D_n f(x) = n(B_n f(x) - f(x))$.

For this approximation Draganov proves in Theorem 6.1 a strong direct and weak converse Voronovskaya inequalities, and as a consequence establishes the characterization

$$\|D_n f - \mathcal{D}f\| = O(n^{-\alpha}) \iff K_{2,\varphi}(f'', t)_{\varphi^2} = O(t^\alpha)$$

under the assumption $f \in W_\infty^2(\varphi)[0, 1]$, $0 < \alpha < 1$.

I highly appreciate the results included in the DSci Thesis of Dr. Borislav Draganov. He shows an excellent awareness of the state of the art of approximation by linear operators. In the proofs of his results he implements techniques and results from classical sources such as the monographs of Ditzian and Totik [22] and of DeVore and Lorentz [18], as well as (for the converse inequalities) of the fundamental paper of Ditzian and Ivanov [23]. The concrete realization in the setup of weighted simultaneous approximations is far from trivial and requires, besides knowledge of the methodology, a lot of technical skill and ingenuity, qualities that Dr. Draganov certainly shows in establishing the representations of the derivatives of the

Bernstein operator and in the proofs of embedding inequalities and the various Jackson, Bernstein and Voronovskaya type inequalities. The matching direct and converse approximation theorems proved by Draganov improve on and/or extend contemporary results of prominent approximation theorists such as Totik, Gonska, Zhou, Knoop, Mache, Ivanov, and give the impression of a completeness of the theory of simultaneous approximation by Bernstein operators in the uniform norm with Jacobi weights.

4. Approbation of the results

The dissertation of Dr. Draganov is based on nine articles published over the past 10 years. Six of these articles are in prestigious scientific journals with Impact factor: three in *J. Approx. Theory*, two in *Results Math.* and one in *Stud. Univ. Babeş-Bolyai Math.* The other three are: one in the Annual of Sofia University and two in the proceedings of international conferences on constructive theory of functions. With only two of his publications ([27] and [29] from the list of references in the dissertation) Draganov exceeds the minimum national requirements under Art. 2b, para. 2 and 3 of ADASRB* for Doctor of Science Degree in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field: 4.5. Mathematics, Indicator B (scoring 135 pts, with a threshold of 100pts). Only two of these papers are co-authored, with Ivan Gadjev and Kamen Ivanov, respectively. I am convinced that Dr. Draganov's contribution to the joint articles is equal to that of his co-authors (as I. Gadjev declared, by the way).

For the articles numbered [25] to [33] on which the dissertation is based, the candidate has found a total of 15 citations, of which three in a monograph published by Birkhäuser, four in Web of Science and one in Scopus, the rest are in preprints, Master Theses and unpublished manuscripts. According to the evaluation methodology, these citations carry 52 points under Indicator D. Dr. Draganov is the author of a total of 38 scientific publications, of which 24 were published after his habilitation. In the attached documents, the applicant indicates 6 more citations (four in WoS and two in Scopus) of his paper in *J. Approx. Theory* from 2010, received after his habilitation, with which he satisfies the minimum requirements for a "Doctor of Science" under Indicator D. A cursory look at Web of Science shows that even some of Dr. Draganov's recent publications have been echoed in the works of other authors, for example, [28] and [36] have two citations each, and [34] has one citation (numbering is from the candidate's CV).

The results presented in his dissertation work and the publications on which it is based have been reported by Dr. Draganov at prestigious international forums in Spain, Poland, France, Austria, Hungary, Ukraine, Turkey, as well as at conferences on constructive theory of functions held in Bulgaria.

I find that the applicant's scientific works exceed the minimum national requirements under Art. 2b, para. 2 and 3 of ADASRB* for Doctor of Science Degree in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field: 4.5. Mathematics. The applicant's results submitted here have not been presented by the candidate in previous procedures for acquiring a scientific title and/or academic position. There is no proven plagiarism in the submitted dissertation and scientific works under this procedure.

5. Qualities of the abstract

The abstract is submitted in both Bulgarian and English, it is 35/36 pages long, with 55 titles cited therein. It gives a complete description and correctly presents the results and content of the dissertation work.

6. Critical notes and recommendations

I have no critical comments on the peer-reviewed dissertation and accompanying papers and documents. The dissertation is written in a very good style (similar to the style in the candidate's book *Measure Theory* for which I had the pleasure of being a reviewer). As a recommendation, I believe that the dissertation could be published as a monograph by a reputable publisher, being a useful resource for approximation theory specialists.

7. Conclusion

Having become acquainted with the DSci Thesis presented in the procedure and the accompanying scientific papers and on the basis of the analysis of their importance and the scientific contributions contained therein, **I confirm** that the presented DSci Thesis and the scientific publications to it, as well as the quality and originality of the results and achievements presented in them, meet the requirements of the Act on Development of the Academic Staff in the Republic of Bulgaria, the Rules for its Implementation and the corresponding Rules at the Sofia University "St. Kliment Ohridski" (FMI-SU) for acquisition by the candidate the scientific degree "Doctor of Science" in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field: 4.5. Mathematics. In particular, the candidate meets the minimal national requirements in the professional field and no plagiarism has been detected in the scientific papers submitted for the competition.

Based on the above, I strongly recommend the scientific jury to award Dr. Borislav Radkov Draganov the scientific degree „Doctor of Science” in the Scientific field 4. Natural Sciences, Mathematics and Informatics, Professional field: 4.5. Mathematics.

Date: February 22, 2024

Signature:

/Geno Nikolov, Professor DSci/